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On Rayleigh's criterion for slowly varying flows

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The inviscid centrifugal instability of slowly varying flows is shown to be asymptotically associated with the lowest-order spatial dependence of the basic flow satisfying Rayleigh's criterion. This result requires special attention for basic flows which reverse their direction. At the instant of reversal the growth rate of the disturbance bifurcates and the choice of the proper branch requires that viscous effects be taken into account within a conveniently small neighbourhood of the branch point. Previous results by Rosenblat (1968) are shown to be incorrect. Such results were based on overlooking the need for viscous effects to be accounted for within a neighbourhood of the bifurcation point. This led to a wrong choice of the path to be followed at bifurcation.

1. Introduction

The aim of the present paper is twofold. First, it is intended to examine the centrifugal instability of unsteady flows for large values of the Taylor number T. The idea is to extend Rayleigh's criterion, in an asymptotic sense, to a class of such flows. Attention is focused on slow variation with time. This is because an extension of the above criterion appears to be relevant under those unsteady conditions where the overall stability is essentially controlled by an integrated effect of the instantaneous stability characteristics. The analysis is specially aimed at the case of slowly varying periodic flows with zero mean. These flows, represented by series expansions in powers of a 'small' characteristic frequency, reduce at lowest order to the class of flows called 'rigid-body oscillations' by Rosenblat (1968). The stability analysis performed by this author led him to conclude that 'rigid-body oscillations' are always linearly and inviscidly stable. However he pointed out that his analysis cannot be expected to hold at low frequencies owing to the large growth which disturbances undergo during the growing part of the cycle, which takes the instability process outside the range of validity of a linear theory. The above results were then used by Rosenblat (1968) to study the inviscid centrifugal instability of more general unsteady flows by means of perturbation expansions based on the 'rigid-body oscillation' case.

Rosenblat's (1968) analysis and his conclusions about the linear inviscid stability of 'rigid-body oscillations' will be shown to be different from the present results. Indeed we find that Rayleigh's criterion applied to the lowest-order spatial dependence of slowly varying cylinder flows essentially controls their instability. The discrepancy between the above conclusions and Rosenblat's (1968) findings seems to be due not to the role of nonlinear effects but rather to an improper choice of the branch of the eigenvalue (the growth rate of the disturbance) at the branch point which occurs when the basic flow reverses its direction. Such a choice cannot be made within the inviscid scheme. Consideration of the full viscous problem is required within a convenient neighbourhood of the branching instant $\bar{\tau}$, where τ is a non-dimensional time variable based on the characteristic period of the basic flow. The matching between the inviscid and viscous solutions determines the proper path to be chosen.

Second, the above ideas are applied to the analysis of the linear stability of purely oscillatory slowly varying flows between coaxial cylinders. The inner (viscous) solution is obtained in much the same way as in Seminara & Hall (1975).

2. Analysis

Consider the flow of an incompressible fluid in the region $R_1 \leqslant r \leqslant R_2$, $0 \leqslant \theta \leqslant \pi$, $-\infty < Z < \infty$, with (r, θ, Z) cylindrical polar co-ordinates and $r = R_1$ and $r = R_2$ rigid walls. Let (U^*, V^*, W^*) denote the corresponding velocity vector, ν the kinematic viscosity and t^* time (a star will denote dimensional quantities).

Let us examine purely azimuthal basic velocity fields described by the velocity vector $(0, \bar{V}^*(r, t^*), 0)$. Furthermore let us assume the flow to be forced by external causes (the rotation of either wall around the axis of the cylinders, or the action of an azimuthal pressure gradient, or some combination of the two) whose time dependence is described by a continuous function $\mathscr{F}(\omega t^*)$ with

$$\sigma \equiv \omega(R_2 - R_1)^2/\nu \ll 1. \tag{1}$$

Under such conditions $\bar{V}^*$ can be given the following asymptotic representation in terms of σ:

$$\bar{V}^* = V_0^* \sum_{n=0}^{\infty} \mathscr{V}_n(\zeta) \mathscr{F}_n(\omega t^*) \sigma^n, \tag{2}$$

where V_0^* is a characteristic speed and

$$\zeta = (r - R_1)/(R_2 - R_1), \quad \mathscr{F}_0(\omega t^*) = \mathscr{F}(\omega t^*). \tag{3}$$

The set of functions $\mathscr{V}_n(\zeta)$ can be obtained by solving a system of ordinary differential equations defined in the interval $(0, 1)$ with suitable boundary conditions.

Let us now consider the linear stability of such basic velocity fields against three-dimensional rotationally symmetric disturbances described by the velocity vector (u^*, v^*, w^*), which we scale such that

$$(u^*, v^*, w^*) = (\nu/2d, V_0^*, \nu/2d)(\tilde{u}, \tilde{v}, \tilde{w}) \quad (d = R_2 - R_1). \tag{4}$$

Furthermore let us restrict ourselves to a class of disturbances amenable to a classical normal-mode analysis. Thus let us set

$$(\tilde{u}, \tilde{v}, \tilde{w}) = \frac{1}{2}\int_{-\infty}^{\infty} \{(u, v, w)\, e^{iaz} + \text{c.c.}\}\, da, \tag{5}$$

where c.c. denotes 'complex conjugate',

$$z = Z/d \tag{6}$$

and a is a non-dimensional wavenumber. If the disturbed flow is substituted into the equations of motion written in non-dimensional form, after linearization and the usual manipulations we obtain

$$\left\{M-\sigma\frac{\partial}{\partial\tau}\right\}Mu-a^2T\left\{\sum_{n=0}^{\infty}\mathscr{V}_n(\zeta)\mathscr{F}_n(\tau)\sigma^n\right\}v=0, \tag{7a}$$

$$\left\{M-\sigma\frac{\partial}{\partial\tau}\right\}v-\frac{1}{2}\left\{\sum_{n=0}^{\infty}\mathscr{D}\mathscr{V}_n\mathscr{F}_n\sigma^n\right\}u=0, \tag{7b}$$

$$u=\partial u/\partial\zeta=v=0\quad(\zeta=0,1), \tag{7c}$$

where

$$\tau=\omega t^*,\quad \delta=(R_2-R_1)/R_1,\quad T=4\{V_0^*(R_2-R_1)/\nu\}^2\delta, \tag{8a–c}$$

$$\overline{\mathscr{D}}=\frac{\partial}{\partial\zeta}+\frac{\delta}{1+\delta\zeta},\quad M=\frac{\partial}{\partial\zeta}\overline{\mathscr{D}}-a^2 \tag{8d, e}$$

and $\mathscr{D}$ is the operator $\overline{\mathscr{D}}$ with $\partial/\partial\zeta$ replaced by $d/d\zeta$.

'Inviscid' (outer) solution

Let us now consider the 'inviscid' limit $T^{\frac{1}{2}}\to\infty$ with a, σ and δ fixed. Under such conditions an 'inviscid' balance is expected to hold between centrifugal effects and local acceleration. Such a balance requires the perturbations to undergo variations (growth or decay) whose time scales are $O(T^{\frac{1}{2}})$, i.e. much shorter than that associated with the basic flow. Thus taking the limit $T^{\frac{1}{2}}\to\infty$ makes the basic flow slowly varying in some sense regardless of its actual frequency σ. An asymptotic approach of the WKB type proves convenient. Thus we consider the following expansions:

$$(u,v)=\sum_{n=0}^{\infty}(u_n^{(i)},T^{-\frac{1}{2}}v_n^{(i)})(T^{-\frac{1}{2}})^n\exp\left\{-i\int\frac{\Omega^{(i)}(\tau)\,d\tau}{T^{-\frac{1}{2}}}\right\}. \tag{9}$$

When these expansions are substituted into (7) and like powers of $T^{-\frac{1}{2}}$ are equated, the leading-order problem for $(u_0^{(i)}, v_0^{(i)})$ is found to depend parametrically on τ. Thus its solution can be determined only up to an arbitrary multiplying function of τ, i.e.

$$(u_0^{(i)},v_0^{(i)})=A(\tau)\,(f_0^{(i)}(\zeta;a,\tau,\sigma),g_0^{(i)}(\zeta;a,\tau,\sigma)). \tag{10}$$

Experience of the WKB approach suggests that the 'amplitude function' $A(\tau)$ is determined by the solvability condition for the $O(T^{-\frac{1}{2}})$ inhomogeneous ordinary differential system for (u_1, v_1). This condition can be shown (see Seminara & Hall 1975) to lead to a first-order homogeneous ordinary differential equation for $A(\tau)$ whose solution is of the form $\exp(-\int H(\tau)\,d\tau)$, where $H(\tau)$ is expressed in terms of the pair of functions $(f_0^{(i)},g_0^{(i)})$, its adjoint $(f_0^{+(i)},g_0^{+(i)})$ and the parameters a, τ and σ. Thus the contribution to the solution (9) due to $A(\tau)$ is equivalent to an $O(T^{-\frac{1}{2}})$ correction for the growth rate $-i\Omega^{(i)}$. For the purposes of the present analysis it suffices to limit ourselves to the consideration of the lowest-order effects. Thus the influence of $A(\tau)$ will be neglected in the following.

Furthermore the class of basic flows considered here ($\sigma\ll1$) is such that the perturbations evolve even faster than (9) suggests.

Indeed, as $\sigma\to0$ (7a,b) show that $-i\Omega^{(i)}=O(\sigma^{-1})$. Let us consider, then, the further limit $\sigma\to0$ and expand $(f_0^{(i)}, g_0^{(i)}, -i\Omega^{(i)})$ in the form

$$(f_0^{(i)},g_0^{(i)},-i\Omega^{(i)})=\sum_{n=0}^{\infty}\left(f_{0n}^{(i)}(\zeta;a,\tau),g_{0n}^{(i)}(\zeta;a,\tau),\frac{-i\Omega_n^{(i)}}{\sigma}\right)\sigma^n. \tag{11}$$

On substituting from (11) into the $O(T^{-\frac{1}{2}})^0$ differential problem for $(f_0^{(i)}, g_0^{(i)}, -i\Omega^{(i)})$ and equating powers of order σ^0, we find

$$Mf_{00} + \tfrac{1}{2}a^2\{\mathscr{F}/\Omega_0^{(i)}\}^2 \mathscr{V}_0 \mathscr{D}\mathscr{V}_0 f_{00} = 0, \tag{12a}$$

$$(-i\Omega_0^{(i)})\,g_{00} + \tfrac{1}{2}\mathscr{F}\mathscr{D}\mathscr{V}_0 f_{00} = 0, \tag{12b}$$

$$f_{00} = 0 \quad (\zeta = 0, 1). \tag{12c}$$

The differential system (12) poses an eigenvalue problem for $-i\Omega_0^{(i)}$ parametrically dependent on τ. However, the τ dependence of $-i\Omega_0^{(i)}$ is easily found to be of the form

$$-i\Omega_0^{(i)} = \pm k\mathscr{F}(\tau), \tag{13}$$

where k is the eigenvalue associated with the problem

$$Nf_{00} + (a^2/2k^2)\mathscr{V}_0\mathscr{D}\mathscr{V}_0 f_{00} = 0, \quad f_{00} = 0 \quad (\zeta = 0, 1). \tag{14a, b}$$

Here N is the operator M with $\partial/\partial\zeta$ replaced by $d/d\zeta$.

The eigenvalue problem (14) is of the classical Sturm–Liouville type, which is encountered in the inviscid analysis of steady centrifugal instability. By appealing to standard theorems on the subject, analysis of (14) leads to the classical Rayleigh theorem; i.e. the characteristic values k^2 are all positive if the discriminant $\mathscr{V}_0\mathscr{D}\mathscr{V}_0$ is everywhere negative and vice versa. In the present case it follows that, provided

$$\mathscr{V}_0\mathscr{D}\mathscr{V}_0 < 0, \tag{15}$$

the growth rate at lowest order (i.e. the value of $-i\Omega$ obtained from (13) with k equal to the most unstable characteristic value of (14) is given by the real function $\pm|k|\mathscr{F}(\tau)$. Thus, if $\mathscr{F}(\tau)$ is a positive (negative) function, the solution $+|k|\mathscr{F}(\tau)$ $(-|k|\mathscr{F}(\tau))$ corresponds to unstable perturbations. However, if $\mathscr{F}(\tau)$ has a zero at $\tau = \bar{\tau}$ the solutions branch at that instant and the question follows of which branch is to be chosen. Rosenblat (1968) considered first the case of time dependence such that the integral $\int_0^{2\pi} \mathscr{F}(\tau)\,d\tau$ vanishes and chose the branch $+|k|\mathscr{F}(\tau)$ both for $\tau > \bar{\tau}$ and $\tau < \bar{\tau}$. Such a choice led him to conclude that 'rigid body oscillations' (slowly varying periodic flows with zero mean) are always inviscidly stable within the framework of a linear theory. However, on a more careful examination, the above conclusion appears to be incorrect.

In fact, the choice of the proper branch cannot be made within the inviscid scheme. Indeed as $\tau \to \bar{\tau}$, $\mathscr{F}(\tau) \to 0$. Assuming $\mathscr{F}'(\tau) \neq 0$ it follows that within a neighbourhood of $\bar{\tau}$ such that $|\tau - \bar{\tau}| \sim O(T^{-\frac{1}{2}})$ the centrifugal terms in (7) are of the same order as the viscous terms, i.e. the full viscous problem (7) is to be considered.

The viscous (inner) solution

Let us rescale the time variable in the region $|\tau - \bar{\tau}| = O(T^{-\frac{1}{2}})$ by defining

$$\mathscr{T} = T^{\frac{1}{2}}(\tau - \bar{\tau}), \quad F_n(\mathscr{T}) = T^{\frac{1}{2}}\mathscr{F}_n(\tau). \tag{16a, b}$$

In terms of the inner variable $\mathscr{T}$ the differential system (7) reads

$$\left\{M - (\sigma T^{\frac{1}{2}})\frac{\partial}{\partial \mathscr{T}}\right\} u^{(v)} - a^2 \left\{\sum_{n=0}^{\infty} \mathscr{V}_n F_n(\mathscr{T})\,\sigma^n\right\} v^{(v)} = 0, \tag{17a}$$

$$\left\{M - (\sigma T^{\frac{1}{2}})\frac{\partial}{\partial \mathscr{T}}\right\} v^{(v)} - \frac{1}{2}\left\{\sum_{n=0}^{\infty} \mathscr{D}\mathscr{V}_n F_n(\mathscr{T})\,\sigma^n\right\} u^{(v)} = 0, \tag{17b}$$

$$u^{(v)} = v^{(v)} = \partial u^{(v)}/\partial\zeta = 0 \quad (\zeta = 0, 1), \tag{17c}$$

where $u^{(v)}$ and $T^{-\frac{1}{2}}v^{(v)}$ denote the functions u and v within the viscous region.

Now let $\sigma \to 0$ and $T^{\frac{1}{2}} \to \infty$ simultaneously but in such a way that

$$\sigma T^{\frac{1}{2}} \to 0, \tag{18}$$

i.e. $\sigma \ll T^{-\frac{1}{2}}$. Under these conditions the differential problem (17) is of the kind treated by Seminara & Hall (1975), i.e. the perturbations again vary rapidly with respect to the basic flow and a WKB type of approach can be employed with $\sigma T^{\frac{1}{2}}$ as a small parameter. Further expansion of the leading-order WKB term in powers of σ leads to the following expression:

$$(u^{(v)}, v^{(v)}) = \left\{ \sum_{n=0}^{\infty} (\sigma)^n (u_n^v(\zeta; \mathscr{T}, a); v_n^v(\zeta; \mathscr{T}, a)) + O(\sigma T^{\frac{1}{2}}) \right\} \times \exp\left\{ -\frac{1}{\sigma T^{\frac{1}{2}}} \int i\Omega_0^v(\mathscr{T})\, d\mathscr{T} \right\}. \tag{19}$$

On substituting from (19) into (17) we find at $O(\sigma^0)$,

$$\{M + i\Omega_0^v(\mathscr{T})\} M f_0^v - a^2 \mathscr{V}_0(\zeta) F_0(\mathscr{T}) g_0^v = 0, \tag{20a}$$

$$\{M + i\Omega_0^v(\mathscr{T})\} g_0^v - \tfrac{1}{2}(\mathscr{D}\mathscr{V}_0) F_0(\mathscr{T}) f_0^v = 0, \tag{20b}$$

$$f_0^v = g_0^v = \partial f_0^v / \partial \zeta = 0 \quad (\zeta = 0, 1), \tag{20c}$$

where

$$(u_0^v, v_0^v) = A^v(\mathscr{T}) \quad (f_0^v, g_0^v), \tag{21}$$

with $A^v(\mathscr{T})$ an amplitude function whose contribution will be neglected at lowest order.

Matching of the inner (viscous) solution and the outer (inviscid) solution requires

$$\lim_{\mathscr{T} \to \pm\infty} (f_0^v(\zeta;\ \mathscr{T}, a), g_0^v(\zeta; \mathscr{T}, a)) = (f_{00}(\zeta;\ a), g_{00}(\zeta;\ a)). \tag{22}$$

Furthermore, since

$$\lim_{\tau \to \bar{\tau}} (-i\Omega_0^{(i)}(\tau)) = \pm k\mathscr{F}'(\bar{\tau})(\tau - \bar{\tau}), \tag{23}$$

the further matching requirement

$$\lim_{\mathscr{T} \to \pm\infty} |-i\Omega_0^v(\mathscr{T})| = |k\mathscr{F}'(\bar{\tau})\mathscr{T}| \tag{24}$$

follows. Conditions (22) and (24) will be proved in the next paragraph by considering the asymptotic form of the solution of system (20) as $\mathscr{T} \to \pm\infty$.

Coming to the crucial point of deciding the branch to be chosen at $\tau = \bar{\tau}$, we first notice an important feature of the differential system (20): at times $\mathscr{T}_1$ and $\mathscr{T}_2$ such that $F_0(\mathscr{T}_1) = -F_0(\mathscr{T}_2)$ it admits identical sets of eigenvalues, i.e.

$$-i\Omega_0^v(\mathscr{T}_1) = -i\Omega_0^v(\mathscr{T}_2), \tag{25}$$

as long as the eigenfunctions satisfy either

$$(f_0^v, g_0^v)_{\mathscr{T}_1} = (-f_0^v, g_0^v)_{\mathscr{T}_2} \quad \text{or} \quad (f_0, g_0)_{\mathscr{T}_1} = (f_0^v, -g_0^v)_{\mathscr{T}_2}. \tag{26a, b}$$

The former alternative (26*a*) requires for continuity that f_0^v should go through zero at $\mathscr{T} = 0$, while the latter requires similar behaviour for g_0. Both alternatives correspond to possible solutions of the differential system (20) with $F_0(\mathscr{T}) = 0$.

The above feature of (20) restricts the possible paths to the following two: $-i\Omega_0^{(i)}$ remains either non-negative or non-positive through $\bar{\tau}$. The choice between these paths requires that the solutions of system (20) should be examined. It proves convenient at this stage to refer to a particular cylindrical flow.

3. Purely oscillatory slowly varying Taylor instability

Let us consider basic flows such that

$$\bar{V}^* = \Lambda R_1 \sum_{n=0}^{\infty} \mathscr{V}_n(\zeta)\mathscr{F}_n(\tau)\,\sigma^n, \tag{27}$$

where

$$\mathscr{V}_0(\zeta) = 1-\zeta, \quad \mathscr{F}(\tau) = \cos\tau \tag{28a, b}$$

and

$$\left.\begin{aligned} &d^2\mathscr{V}_{n+1}/d\zeta^2 = \mathscr{V}_n(\zeta), \quad \mathscr{F}_n(\tau) = d^n\mathscr{F}/d\tau^n \\ &\mathscr{V}_n = 0 \quad (\zeta = 0, 1) \end{aligned}\right\} \quad (n = 0, 1, 2, \ldots). \tag{29a, b, c}$$

The relationships (27)–(29) give the basic velocity field produced in a viscous fluid by an inner cylinder performing harmonic oscillations about its axis with angular speed Λ when $\delta/R_1 \ll 1$ can be considered a valid approximation. The differential system governing the disturbed flow is still (7) with T given by (8c), V_0^* equal to ΛR_1 and

$$\tilde{\mathscr{D}} \equiv \partial/\partial\zeta, \quad M = \partial^2/\partial\zeta^2 - a^2. \tag{30a, b}$$

The growth rate in the outer inviscid region is given by (13). The eigenrelation associated with the eigenvalue problem (14) with $\mathscr{V}_0$ given by (28a) is well known (see Chandrasekhar 1961, p. 290). We found it more convenient to solve the eigenvalue problem (14) numerically by means of a fourth-order Runge–Kutta integration scheme with 40 steps. The results obtained for the most unstable eigenrelation $k = k(a^2)$ confirm the known results. In particular $k(10)$ was found to be equal to 0·368415.

The (inner) viscous solution is governed by the differential system (20) with $\mathscr{V}_0$ and $\mathscr{F}$ given by (28a, b) respectively. We obtained the asymptotic solutions of this system as $\mathscr{T} \to 0$ and $\mathscr{T} \to \pm\infty$. These asymptotic results were then extended by performing a numerical integration of system (20) in the range $\mathscr{T} = 0$–10^3.

Let us first examine the asymptotic behaviour of the solutions of system (20) as $\mathscr{T} \to 0$. One set of solutions is associated with each of the alternatives (26a, b). We consider the two sets separately.

(i) *The case in which f_0^v vanishes at $\tau = \bar{\tau}$.* The structure of the differential system (20) suggests that the following expansions hold for 'small' $\mathscr{T}$:

$$f_0^v = f_{01}^v\mathscr{T} + O(\mathscr{T}^3), \tag{31a}$$

$$g_0^v = g_{00}^v + g_{02}^v\mathscr{T}^2 + O(\mathscr{T}^4), \tag{31b}$$

$$\Omega_0^v = \Omega_{00}^v + \Omega_{02}^v\mathscr{T}^2 + O(\mathscr{T}^4). \tag{31c}$$

Substituting from (31) into (20) and equating terms $O(\mathscr{T}^0)$ gives

$$(N + i\Omega_{00}^v)\,g_{00}^v = 0, \tag{32a}$$

$$g_{00}^v = 0 \quad (\zeta = 0, 1). \tag{32b}$$

Thus

$$g_{00}^v = \sin m\pi\zeta \quad (m = 1, 2, 3, \ldots), \tag{33}$$

$$i\Omega_{00}^v = a^2 + m^2\pi^2, \tag{34}$$

where the arbitrary constant in (33) has been put equal to one.

Substitution of (31) into (20) leads at $O(\mathscr{T})$ to the following inhomogeneous ordinary differential system for f_{01}^v:

$$(N+i\Omega_{00}^v)\,Nf_{01}^v = -a^2\mathscr{V}_0\,g_{00}^v, \tag{35a}$$

$$f_{01}^v = df_{01}^v/d\zeta = 0 \quad (\zeta = 0, 1). \tag{35b}$$

The solution of (35) can be written in the form

$$f_{01}^v = (\alpha+\mu\zeta)\sin m\pi\zeta + (\beta+\lambda\zeta+\nu\zeta^2)\cos m\pi\zeta + \delta\sinh a\zeta + \epsilon\cosh a\zeta, \tag{36}$$

where

$$\lambda = -\{2\rho a(1+\rho^2)\}^{-1}, \quad \mu = -(1+5\rho^2)\,\lambda^2, \quad \nu = -\tfrac{1}{2}\lambda, \tag{37a, b, c}$$

$$\alpha = \lambda\{-(2a\rho)^{-1}+\tfrac{1}{2}(1+5\rho^2)\,\lambda+\tau/4\rho\}, \quad \delta = -(\lambda/a+\rho\alpha) \tag{37d, e}$$

$$\beta = \tfrac{1}{2}\lambda\{-[a^{-1}+(1+5\rho^2)\,\rho\lambda]\,\tau-\tfrac{1}{2}\}, \quad \epsilon = -\beta, \tag{37f, g}$$

and

$$\rho = \frac{m\pi}{a}, \quad \tau = \frac{\sinh a}{\cosh a-(-1)^m}. \tag{38}$$

In order to determine the $O(\mathscr{T}^2)$ correction for the growth rate we need to examine the $O(\mathscr{T}^2)$ system obtained on substituting (31) into (20). We find

$$(N+i\Omega_{00}^v)\,g_{02}^v = -i\Omega_{02}^v\,g_{00}^v+\tfrac{1}{2}f_{01}^v, \tag{39a}$$

$$g_{02}^v = 0 \quad (\zeta = 0, 1). \tag{39b}$$

One can readily show that a certain condition has to be satisfied for the above system to admit a solution. Since the homogeneous operator associated with (39a) is self-adjoint this solvability condition reads

$$-i\Omega_{02}^v = -\int_0^1 f_{01}^v\,g_{00}\,d\zeta. \tag{40}$$

Substituting from (32) and (36) into (40) and integrating, we find

$$-i\Omega_{02}^v = -\frac{\lambda}{8\rho}\left(\frac{1}{a}+\tau-\frac{4a}{a^2+m^2\pi^2}\right). \tag{41}$$

This expression shows that $-i\Omega_{02}^v$ is positive for any m and a. In particular, for $m = 1$ (the most unstable eigenvalue) and $a^2 = 10$ we obtain

$$-i\Omega_0^v = -19{\cdot}869604+0{\cdot}006031\mathscr{T}^2+O(\mathscr{T}^4). \tag{42}$$

The sign of $-i\Omega_{02}^v$ suggests that the growth rate associated with the most unstable viscous solution of set (i) tends towards the positive branches of the inviscid growth rate at least for small (positive or negative) values of $\mathscr{T}$. The numerical results to be discussed and the asymptotic solution of (20) for $\mathscr{T}\to\pm\infty$ will indeed confirm that the above behaviour persists as $\mathscr{T}$ increases.

(ii) *The case in which g_0^v vanishes at $\tau = \bar{\tau}$.* Let us now expand the solution of (20) in the form

$$f_0^v = f_{00}^v+f_{02}^v\mathscr{T}^2+O(\mathscr{T}^4), \tag{43a}$$

$$g_0^v = g_{01}^v\mathscr{T}+O(\mathscr{T}^3), \tag{43b}$$

$$\Omega_0^v = \Omega_{00}^v+\Omega_{01}^v\mathscr{T}^2+O(\mathscr{T}^4). \tag{43c}$$

The $O(\mathscr{T}^0)$ system obtained after substituting (43*a*, *b*, *c*) into (20) reads

$$\{N+i\Omega^v_{00}\}Nf^v_{00} = 0, \tag{44a}$$

$$f^v_{00} = df^v_{00}/d\zeta = 0 \quad (\zeta = 0, 1). \tag{44b}$$

This eigenvalue problem was solved by Dolph & Lewis (1958), who found two sets of eigenvalues and associated eigenfunctions. The more unstable of these corresponds to even eigenfunctions and leads to the following solutions of (44):

$$(i\Omega^v_{00})_e = a^2+4x_k^2, \tag{45a}$$

$$(f^v_{00})_e = \cos x_k(2\zeta-1)-\frac{\cos x_k}{\cosh \frac{1}{2}a}\cosh \tfrac{1}{2}a(2\zeta-1), \tag{45b}$$

where x_k is a root of the transcendental equation

$$x_k \tan x_k = -\tfrac{1}{2}a \tanh \tfrac{1}{2}a, \tag{46}$$

and the arbitrary constant in (45*b*) has been put equal to 1. The second sets of eigenvalues of (44) corresponds to odd eigenfunctions. These are given by

$$(i\Omega^v_{00})_0 = a^2+4y_k^2, \tag{47a}$$

$$(f^v_{00})_0 = \sin y_k(2\zeta-1)-\frac{\sin y_k}{\sinh \frac{1}{2}a}\sinh \tfrac{1}{2}a(2\zeta-1), \tag{47b}$$

where y_k is a root of the transcendental equation

$$\frac{\tan y_k}{y_k} = \frac{\tanh \frac{1}{2}a}{\frac{1}{2}a}. \tag{48}$$

We shall consider in the following only the even solutions (45). In fact they correspond to the more unstable disturbances at $\mathscr{T} = 0$ and some numerical results to be presented show that this behaviour persists as $\mathscr{T}$ increases.

Substitution of (43) into (20) leads at $O(\mathscr{T})$ to the following inhomogeneous differential problem:

$$\{N+i\Omega^v_{00}\}g^v_{01} = \tfrac{1}{2}f^v_{00}, \tag{49a}$$

$$g^v_{01} = 0 \quad (\zeta = 0, 1). \tag{49b}$$

This system, with $i\Omega^v_{00}$ and f^v_{00} given by (45*a*, *b*) respectively, can be easily solved to give

$$\begin{aligned} g^v_{01} = {} & (2i\Omega^v_{00})^{-1}\cos x_k(2\zeta-1)-(16x_k\cos x_k)^{-1}\sin 2x_k\zeta \\ & +(8x_k)^{-1}\zeta\sin x_k(2\zeta-1)-(2i\Omega^v_{00}\cosh\tfrac{1}{2}a)^{-1}\cos x_k\cosh\tfrac{1}{2}a(2\zeta-1). \end{aligned} \tag{50}$$

The evaluation of the $O(\mathscr{T}^2)$ correction to $i\Omega^v_0$ requires that the procedure be continued to the next order. We find

$$(N+i\Omega^v_{00})Nf^v_{02} = (-i\Omega^v_{02})Nf^v_{00}-a^2\mathscr{V}_0 g^v_{01}, \tag{51a}$$

$$f^v_{02} = df^v_{02}/d\zeta = 0 \quad (\zeta = 0, 1). \tag{51b}$$

The solvability condition required for the system (51*a*, *b*) to admit a solution defines $-i\Omega^v_{02}$ in the form

$$-i\Omega^v_{02} = a^2\int_0^1 \mathscr{V}_0 g^v_{01}f^v_{00}\,d\zeta \Big/ \int_0^1 f^v_{00}\,Nf^v_{00}\,d\zeta, \tag{52}$$

where use has been made of the self-adjointness of the homogeneous operator associated with (51a). Substituting from (45) and (50) into (52) and performing the integrations leads after much tedious algebra to

$$-i\Omega_{02}^{v} = -\frac{4a^2 x_k}{i\Omega_{00}^{v}(2x_k+\sin 2x_k)}\left\{\frac{1}{4i\Omega_{00}^{v}}+\frac{\sin 2x_k}{8x_k i\Omega_{00}^{v}}-\frac{1}{128x_k^2}\right.$$
$$\left.+\frac{\sin 3x_k}{512x_k^3\cos x_k}+\frac{\tan x_k}{512x_k^3}(1-8x_k^2)+\frac{\cos^2 x_k(a+\sinh a)}{8ai\Omega_{00}^{v}\cosh^2\frac{1}{2}a}\right\}. \quad (53)$$

In the appendix we prove that $-i\Omega_{02}^{v}$ is negative for any a. In particular, the most unstable eigenvalue with $a^2 = 10$ reads

$$-i\Omega_0^v = -37{\cdot}84172 - 0{\cdot}00589984\mathscr{T}^2 + O(\mathscr{T}^4). \quad (54)$$

The form of $-i\Omega_0^v$ suggests that the growth rate associated with the most unstable viscous solution of set (ii) tends, for small (positive or negative) values of $\mathscr{T}$, towards the negative branches of the inviscid growth rate. The numerical results presented below will show that this behaviour does not persist for large values of $\mathscr{T}$. However the most unstable eigenvalue of set (ii) remains more stable than the most unstable eigenvalue of set (i), which is then the one relevant for our analysis.

We now look for an asymptotic solution of (20) (with $\mathscr{V}_0$ and $\mathscr{F}$ given by (28a, b) respectively) as $\mathscr{T} \to \pm\infty$. This analysis will allow the inner solution to be matched to the outer solution. Furthermore it will provide a useful check on the numerical results to be presented.

Three flow regions can be distinguished as $|\mathscr{T}| \to \infty$: an inviscid (central) region where $\zeta = O(1)$, an inner viscous layer adjacent to the inner wall where $\zeta = O(\pm\mathscr{T})^{-\frac{1}{2}}$ and an outer viscous layer adjacent to the outer wall where $1-\zeta = O(\pm\mathscr{T})^{-\frac{1}{2}}$. (The $\pm$ sign corresponds to $\mathscr{T} \gtrless 0$.) Now for the set of branching points such that $\mathscr{F}'(\bar{\tau}) = +1$ $(\bar{\tau} = \frac{1}{2}(4m-1)\pi,\ m = 0, \pm 1, \pm 2, \ldots)$, $F_0 \sim \mathscr{F}$ as $\mathscr{T} \to \pm\infty$. Thus the structure of (20) and (26a) suggests that the variables be redefined in the central region in the form

$$\Omega_0^v = (\pm\mathscr{T})\sigma, \quad f_0^v = \pm\varphi, \quad g_0^v = \gamma. \quad (55a, b, c)$$

On substituting (55) into (20) we find the following differential problem for (ϕ, γ, σ):

$$i\sigma N\varphi - a^2(1-\zeta)\gamma = -(\pm\mathscr{T})^{-1}N^2\varphi, \quad (56a)$$

$$i\sigma\gamma + \tfrac{1}{2}\varphi = -(\pm\mathscr{T})^{-1}N\gamma \quad (56b)$$

with suitable boundary conditions. Let us now expand (ϕ, γ, σ) in the form

$$(\varphi,\gamma,\sigma) = (\varphi_0,\gamma_0,\sigma_0)+(\varphi_1,\gamma_1,\sigma_1)(\pm\mathscr{T})^{-\frac{1}{2}}+(\varphi_2,\gamma_2,\sigma_2)(\pm\mathscr{T})^{-1}+O(\pm\mathscr{T}^{\frac{3}{2}}). \quad (57)$$

On substituting (57) into (56) we find at leading order an eigenproblem for (φ_0, σ_0) identical to (14) with $\mathscr{V}_0$ given by (28a), $f_{00}^{(i)}$ replaced by φ_0, and σ_0 replaced by $(\pm ik)$. The solution,

$$i\sigma_0 = -k, \quad \varphi_0 = f_{00}^{(i)}, \quad \gamma_0 = \tfrac{1}{2}\varphi_0, \quad (58a, b, c)$$

corresponds to unstable disturbances for both positive and negative $\mathscr{T}$.

Before proceeding to higher-order approximations for the solution in the central region we need to consider the flow in the wall viscous layers. Let us rescale the variables in the inner viscous layer such that

$$\xi = (\pm\mathscr{T})^{\frac{1}{2}}\zeta, \quad f_0^v = \pm(\pm\mathscr{T})^{-\frac{1}{2}}\phi, \quad g_0^v = (\pm\mathscr{T})^{-\frac{1}{2}}\Gamma. \quad (59a, b, c)$$

Substitution of (59) into (20) gives

$$\{\partial^2/\partial\xi^2+i\sigma-a^2(\pm\mathscr{T})^{-1}\}\{\partial^2/\partial\xi^2-a^2(\pm\mathscr{T})^{-1}\}\phi=(\pm\mathscr{T})^{-1}a^2\{1-(\pm\mathscr{T})^{-1}\xi\}\Gamma, \quad (60a)$$

$$\{\partial^2/\partial\xi^2+i\sigma-a^2(\pm\mathscr{T})^{-1}\}\Gamma+\tfrac{1}{2}\phi=0, \quad (60b)$$

$$\phi=d\phi/d\xi=\Gamma=0 \quad (\xi=0). \quad (60c)$$

As $\xi\rightarrow\infty$ (ϕ,Γ) must behave such that the inner-layer solution matches the limiting form of the solution in the central region as $\zeta\rightarrow 0$. Let us expand

$$(\phi,\Gamma)=(\phi_0,\Gamma_0)+(\phi_1,\Gamma_1)(\pm\mathscr{T})^{-\frac{1}{2}}+O(\pm\mathscr{T})^{-1}. \quad (61)$$

The leading-order problem for (ϕ_0,Γ_0) is easily found from (61), (60) and (56) and reads

$$(d^2/d\xi^2-k)\,d^2\phi_0/d\xi^2=0, \quad (62a)$$

$$(d^2/d\xi^2-k)\,\Gamma_0=-\tfrac{1}{2}\phi_0, \quad (62b)$$

$$\phi_0=d\phi_0/d\xi=\Gamma_0=0 \quad (\xi=0). \quad (62c)$$

This system is solved subject to the matching conditions for $\xi\rightarrow\infty$. We find

$$\phi_0=k^{-\frac{1}{2}}\{\exp[-(k\xi)^{\frac{1}{2}}]-1\}+\xi, \quad (63a)$$

$$\Gamma_0=(2k^{\frac{3}{2}})^{-1}\{\exp[-(k\xi)^{\frac{1}{2}}]-1\}+(2k)^{-1}\xi\{1+0{\cdot}5\exp[-(k\xi)^{\frac{1}{2}}]\}, \quad (63b)$$

where exponentially increasing terms have been rejected and ϕ_0 has been normalized such that $(d\phi_0/d\zeta)_{\zeta=0}=1$. The solution (63*a*) implies that in the central region

$$\phi\sim\zeta-k^{-\frac{1}{2}}(\pm\mathscr{T})^{-\frac{1}{2}}+O(\pm\mathscr{T})^{-1} \quad (\zeta\rightarrow 0). \quad (64)$$

The flow in the outer viscous layer can be similarly studied after the rescaling

$$\eta=(\pm\mathscr{T})^{\frac{1}{2}}(1-\zeta),\quad f_0^v=\pm(\pm\mathscr{T})^{-\frac{1}{2}}\psi,\quad g_0^v=(\pm\mathscr{T})^{-\frac{1}{2}}K \quad (65\,a,b,c)$$

and is governed by a differential problem identical to (60) with ϕ replaced by ψ, Γ by K and $\{1-(\pm\mathscr{T})^{-\frac{1}{2}}\xi\}$ by $(\pm\mathscr{T})^{-\frac{1}{2}}\eta$. If we now expand (ψ,K) in the form

$$(\psi,K)=(\psi_0,K_0)+(\psi_1,K_1)(\pm\mathscr{T})^{-\frac{1}{2}}+O(\pm\mathscr{T})^{-1} \quad (66)$$

the leading-order solution for (ψ_0,K_0) is readily found to be identical to that for (ϕ_0,Γ_0) but with a factor D equal to $(-d\varphi_0/d\zeta)_{\zeta=1}$, which is required for the matching with the central-region solution. Thus it follows that

$$\phi\sim D(1-\zeta)-Dk^{-\frac{1}{2}}(\pm\mathscr{T})^{-\frac{1}{2}}+O(\pm\mathscr{T})^{-1} \quad (\zeta\rightarrow 1). \quad (67)$$

Let us now return to the flow in the central region. If we equate terms $O(\pm\mathscr{T})^{-\frac{1}{2}}$ in (56) and use (64) and (67) we find

$$N\varphi_1+\frac{a^2(1-\zeta)}{2k^2}\varphi_1=-\frac{i\sigma_1}{k^3}a^2\mathscr{V}_0\varphi_0, \quad (68a)$$

$$\varphi_1=-1/k^{\frac{1}{2}}\quad(\zeta=0),\quad \varphi_1=-D/k^{\frac{1}{2}}\quad(\zeta=1). \quad (68b,c)$$

The above system admits a solution provided that the following condition is satisfied:

$$-i\sigma_1=-k^{\frac{5}{2}}(1+D^2)/a^2\int_0^1\mathscr{V}_0\varphi_0^2\,d\zeta, \quad (69)$$

which shows that $-i\sigma_1$ is negative for any value of a. The solution of (68) is unique but for an arbitrary multiple of the eigenfunction φ_0. However it will appear that

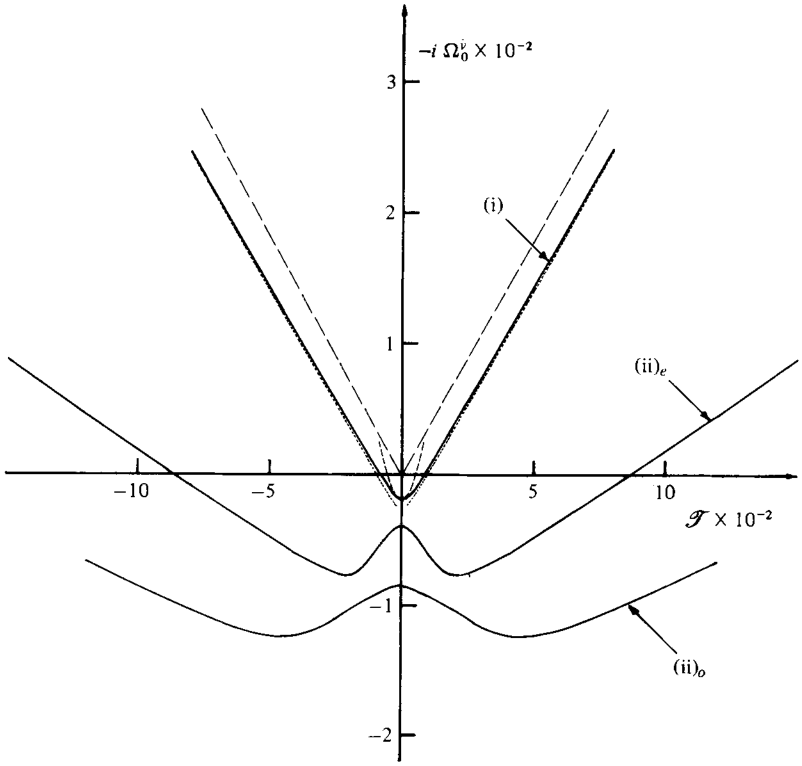

FIGURE 1. The numerical results for the most unstable eigenvalues in the viscous region [solid curve (i)] are shown along with the results of the asymptotic analysis as $\mathscr{T} \to 0$ (dashed curve) and $\mathscr{T} \to \pm\infty$ (dotted curve) for $a^2 = 10$. The unstable branches of the inviscid eigenvalue ($a^2 = 10$) (dashed curve) are also shown in the viscous region along with the numerical results for the most unstable eigenvalues of the set (ii) with even (ii)$_e$ and odd (ii)$_o$ eigenfunctions.

this lack of uniqueness does not influence the solution for the growth rate at higher order.

The above procedure can then be continued. The solution (69) for $-i\sigma_1$ can be used to determine the flow in the viscous layers at second order. The limiting form of the latter at the edge of the layers determines the boundary conditions for the inviscid flow in the central region at third order. The solvability condition for the inhomogeneous differential problem found at this stage determines $-i\sigma_2$. For the sake of brevity we just give the main result of this analysis; the $O(\mathscr{T}^0)$ correction for the growth rate is found to be

$$-i\sigma_2 = -\frac{a^2}{2k^2}\frac{\int_0^1 (\mathscr{V}_0\varphi_0)^2\,d\zeta}{\int_0^1 \mathscr{V}_0\varphi_0^2\,d\zeta} - \frac{(i\sigma_1)^2}{2k} + (i\sigma_1)\frac{\int_0^1 \mathscr{V}_0\varphi_1\varphi_0\,d\zeta}{\int_0^1 \mathscr{V}_0\varphi_0^2\,d\zeta} - \frac{k^{\frac{5}{2}}}{a^2}\frac{B_1 + DB_2 + \frac{1}{2}i\sigma_1 k^{-1}(1+D)}{\int_0^1 \mathscr{V}_0\varphi_0^2\,d\zeta}, \qquad (70)$$

where B_1 and B_2 are given by $(d\varphi_1/d\zeta)_{\zeta=0}$ and $(-d\varphi_1/d\zeta)_{\zeta=1}$ respectively. By using (69) it can be shown that adding any multiple of φ_0 to φ_1 would affect both the third

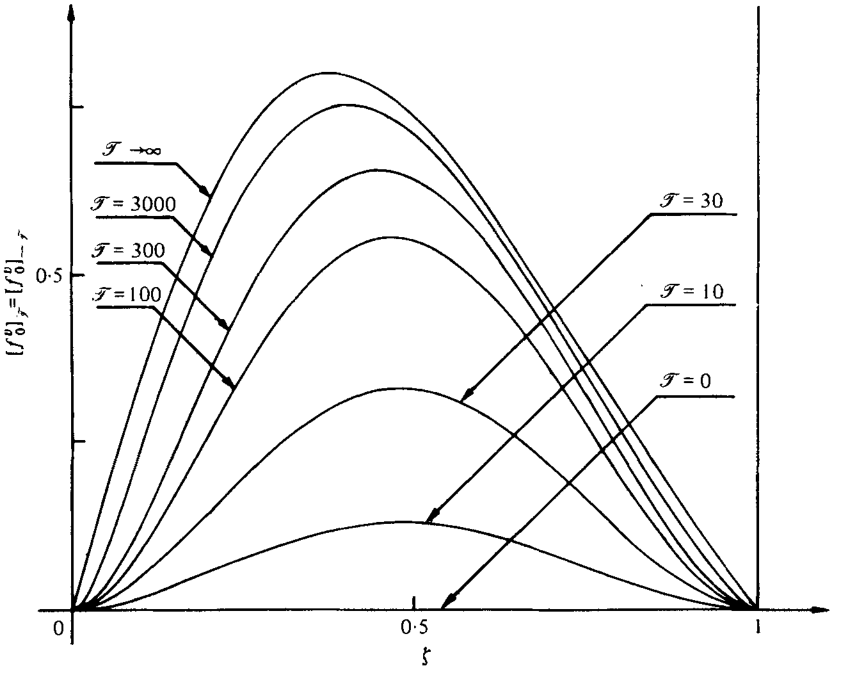

FIGURE 2. The viscous eigenfunction f_0^v normalized such that $(g_0^v)_{\zeta=\frac{1}{2}} = 1$ is shown as a function of $\mathscr{T}$ for $a^2 = 10$. The curve for $\mathscr{T} \to \infty$ is the inviscid eigenfunction $f_{00}^{(i)}$.

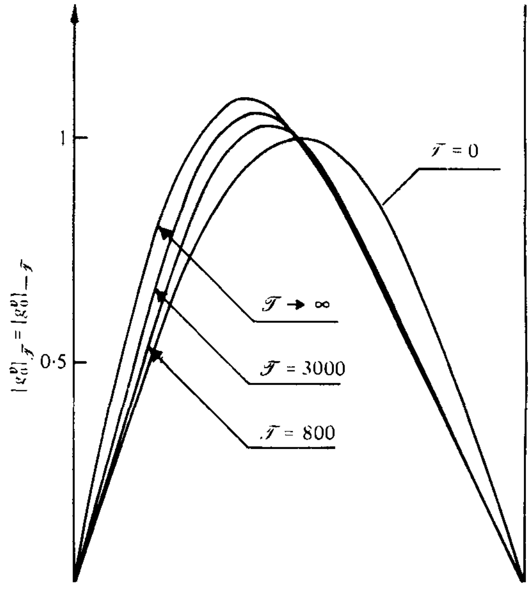

FIGURE 3. The viscous eigenfunction g_0^v normalized such that $(g_0^v)_{\zeta=\frac{1}{2}} = 1$ is shown as a function of $\mathscr{T}$ for $a^2 = 10$. The curve for $\mathscr{T} \to \infty$ is the inviscid eigenfunction $g_{00}^{(i)}$.

$\lvert\mathscr{T}\rvert$	$-i\Omega_0^v$ Asymptotic solution	Numerical solution
0	−19·86960	−19·86961
0·1	−19·86954	−19·86955
0·3	−19·86906	−19·86907
0·5	−19·86810	−19·86811
1	−19·86357	−19·86358
5	−19·7188	−19·7201
10	−19·2665	−19·2851
20	−17·4572	−17·7100
50	−4·7921	−10·0134

TABLE 1. Comparison between numerical and asymptotic results as $\mathscr{T} \to 0$ for the most unstable eigenvalue of set (i) ($a^2 = 10$).

$\lvert\mathscr{T}\rvert$	$-i\Omega_0^v$ Asymptotic solution	Numerical solution
50	−12·992	−10·013
100	3·41	5·72
300	71·68	73·77
500	141·66	143·66
3000	1023·1	1024·8

TABLE 2. Comparison between numerical and asymptotic results as $\mathscr{T} \to \pm\infty$ for the most unstable eigenvalue of set (i) ($a^2 = 10$).

and the fourth term of the right side of (70) in such a way that $-i\sigma_2$ would not be altered.

The asymptotic analysis for large $\lvert\mathscr{T}\rvert$ developed above shows that the matching requirements (22) and (24) are met by the solution (55), (57) and (58) at leading order.

The values of $-i\sigma_1$ and $-i\sigma_2$ were calculated for $a^2 = 10$. Using the eigenfunction φ_0 (normalized as mentioned above) and the corresponding eigenvalue k, we evaluated $-i\sigma_1$ from (69) numerically by means of Simpson's rule. Using the value of $-i\sigma_1$ thus obtained, the inhomogeneous system (68) was solved by the method of linear combinations (see Eagles 1971). A Runge–Kutta integration scheme with 40 steps was used to obtain each independent solution numerically. The solution for φ_1 was finally used to evaluate $-i\sigma_2$ from (70). We found

$$(-i\Omega_0^v) = 0{\cdot}368415(\pm\mathscr{T}) - 0{\cdot}72344(\pm\mathscr{T})^{\frac{1}{2}} - 26{\cdot}297 + O(\pm\mathscr{T})^{-\frac{1}{2}} \quad \text{for} \quad a^2 = 10. \tag{71}$$

The system (20) (with $\mathscr{V}_0$ given by (28*a*), $F_0 = \mathscr{T}$ and $a^2 = 10$) was also solved numerically, again by means of the method of linear combinations and a Runge–Kutta integration scheme with 40 steps. The results for the most unstable eigenvalue $-i\Omega_0^v$ as a function of $\mathscr{T}$ in the interval $-10^3 < \mathscr{T} < 10^3$ are shown in figure 1. The agreement between the numerical solution and the asymptotic solutions for $\mathscr{T} \to 0$ and

$\lvert\mathscr{T}\rvert$	$-i\Omega_0^v$ Asymptotic solution	Numerical solution
0	−37·84172	−37·8419
0·3	−37·84224	−37·8424
0·6	−37·84384	−37·8440
0·8	−37·84549	−37·8457
1	−37·84761	−37·8478
5·1	−37·99517	−37·9941
11·1	−38·56864	−38·5415
15·1	−39·18694	−39·0993
20	−40·20165	−39·9553
40	−47·28146	−44·9443
100	−96·84015	−60·5240

TABLE 3. Comparison between numerical and asymptotic results as $\mathscr{T} \to 0$ for the most unstable eigenvalue of set (ii) (even eigenfunctions, $a^2 = 10$).

$\mathscr{T} \to \pm\infty$ appears to be satisfactory. Tables 1 and 2 below confirm this agreement in more detail. Figures 2 and 3 show the variation with $\mathscr{T}$ of the viscous eigenfunctions (f_0^v, g_0^v) along with the inviscid eigenfunctions $(f_{00}^{(i)}, g_{00}^{(i)})$.

Figure 1 also shows the numerical results obtained for the most unstable eigenvalue corresponding to even eigenfunctions at $\mathscr{T} = 0$ (see discussion of set (ii) above). It appears that the initial trend predicted by the asymptotic solution as $\mathscr{T} \to 0$ [see (54)] and confirmed by the numerical results (see table 3) modifies sharply as $\lvert\mathscr{T}\rvert$ increases. For large $\lvert\mathscr{T}\rvert$ this eigenvalue becomes positive, remaining less unstable than the eigenvalue discussed previously.

Similar results are shown in figure 1 for the most unstable eigenvalue of set (ii), corresponding to odd eigenfunctions. This confirms that the eigenvalue relevant for present purposes is the most unstable of the set corresponding to the condition (26*a*). The analysis performed above shows that as $\mathscr{T} \to \pm\infty$ this eigenvalue matches the positive branches of the most unstable inviscid eigenvalue.

4. Conclusions

The analysis and the results discussed in §§ 2 and 3 show that, in the inviscid limit, the linear stability of the class of flows considered in this paper depends on the sign of $\mathscr{V}_0\mathscr{D}\mathscr{V}_0$. Thus Rayleigh's criterion has been extended to slowly varying flows, at least in an asymptotic sense. In view of (13) this conclusion is trivial for basic flows which do not reverse their direction. It requires more attention in cases where $\mathscr{F}(\tau)$ changes sign. The problem of deciding which branch of the growth rate is to be chosen at bifurcation requires the effect of viscosity to be accounted for within a conveniently small neighbourhood of the branch point. The proper branch is found to be that such that $-i\Omega_0^{(i)}$ remains non-negative through the point $\bar{\tau}$. It is to be emphasized that the inviscid eigenfunction $f_{00}(\zeta;\, a, \tau)$ corresponding to such a choice of branch exhibits a jump discontinuity at $\tau = \bar{\tau}$, which is smoothed out (see figure 2) when viscous effects are properly accounted for. The validity of the above results does not depend on the assumption $\mathscr{F}'(\bar{\tau}) \neq 0$. The order n of the first non-vanishing derivative of $\mathscr{F}$ at

$\tau = \bar{\tau}$ determines the order of magnitude $T^{-\frac{1}{2}n}$ of the viscous region. Moreover the inequality $\sigma \ll T^{-\frac{1}{2}}$ becomes in general

$$\sigma \ll T^{-\frac{1}{2}n}. \tag{72}$$

Thus 'rigid-body oscillations' appear to be always unstable provided that their spatial dependence is such that the criterion (15) is satisfied. Such a conclusion disagrees with that derived by Rosenblat (1968) and invalidates most of his results on the inviscid instability of more general unsteady cylinder flows. In particular, associating the inviscid centrifugal instability of flows which are periodic about a zero mean with the existence of a phase lag between velocity and vorticity does not appear to be a correct criterion. Under such conditions, at least for a slowly varying time dependence, the nature of the instability mechanism does not seem to be different from that which characterizes a steady basic state. The slow variation of the latter leads to only a slow variation of the disturbance growth (or decay) and the integrated effect over a period controls the stability of the basic configuration.

A natural question that arises from the present results is whether it might be worth trying to extend Rayleigh's criterion to unsteady basic flows which are not slowly varying in time. However under the latter conditions the time dependence of the basic flow is not separable in general from its spatial dependence. Thus Rayleigh's criterion is likely to be satisfied within restricted portions of the flow domain which moreover vary relatively fast in time. The relevance of the criterion for such flow configurations does then appear to be dubious even though qualitative results (see Seminara & Hall 1976) referring to high frequency basic flows would tend to support its validity. Further work is required to make the latter arguments more convincing.

The author acknowledges the suggestion of one of the referees that the asymptotic analysis for small $\mathscr{T}$ should be performed in order to substantiate the numerical results.

Appendix

We now prove that the relationship (53) defines $-i\Omega_{02}^{v}$ as a negative number for any (positive) value of a and x_k such that (46) is satisfied. We denote the terms in the braces on the right side of (53) by $\{1\}$, $\{2\}$, $\{3\}$, $\{4\}$, $\{5\}$, $\{6\}$ respectively.

In view of (46), $\tan x_k/x_k^3$ is negative. Moreover $|x_k| > 1$, whence

$$\{5\} > -\frac{7}{512}\frac{\tan x_k}{x_k} > 0. \tag{A 1}$$

Also, by inspection it follows that

$$\{6\} > 0. \tag{A 2}$$

Since $|\sin 2x_k/x_k| < 1$ we find

$$\tfrac{1}{2}\{1\}+\{2\} > 0. \tag{A 3}$$

Using the identity

$$\sin 3x = 3\sin x - 4\sin^3 x,$$

we can write

$$\{4\} = \frac{\tan x_k}{512x_k^3}(3-4\sin^2 x_k) > \frac{3}{512}\frac{\tan x_k}{x_k^3}, \tag{A 4}$$

whence

$$\{4\}+\tfrac{3}{7}\{5\} > 0. \tag{A 5}$$

Finally if $|\tan x_k/x_k| > x_k^{-2}$ it follows that

$$\tfrac{4}{7}\{5\}+\{3\} > 0. \tag{A 6}$$

If $|\tan x_k/x_k| < x_k^{-2}$ then, from (46), a is found to be less than 2. Thus $\frac{1}{2}\{1\}$ is greater than $\frac{1}{64}u_k^2$ and

$$\tfrac{1}{2}\{1\}+\{3\} > 0. \tag{A 7}$$

The inequalities (A 2), (A 3), (A 5), (A 6) and (A 7) prove our statement.

REFERENCES

CHANDRASEKHAR, S. 1961 *Hydrodynamic and Hydromagnetic Stability*. Oxford: Clarendon Press.

DOLPH, C. L. & LEWIS, D. C. 1958 *Quart. Appl. Math.* **16**, 97.

EAGLES, P. M. 1971 *J. Fluid Mech.* **49**, 529.

ROSENBLAT, S. 1968 *J. Fluid Mech.* **33**, 321.

SEMINARA, G. & HALL, P. 1975 *Proc. Roy. Soc.* A **346**, 279.

SEMINARA, G. & HALL, P. 1976 *Proc. Roy. Soc.* A **350**, 299.